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Approximations to Some Generalized Functions; Application to Array Processing

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PREFACE

This research was conducted under NUSC Project No. A75205, Subproject No. ZR0000101, "Applications of Statistical Communication Theory to Acoustic Signal Processing," Principal Investigator Dr. Albert H. Nuttall (Code 3302), Program Manager Mr. Robert M. Hillyer, Naval Material Command (MAT 05).

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Approximations to some generalized functions generated by the process of differentiation are developed and used to interpret some integrals involving generalized functions. Application to the realization of the ideal voltage response pattern of an array is made and illustrated for both an impulsive and a smooth approximation to the required generalized weighting function.

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LIST OF SYMBOLS

$f(x)$	ordinary function
$g(x)$	generalized function
$h(x)$	good function
ϵ	parameter controlling the transition region
$f_\epsilon(x)$	approximation to f(x)
$g_\epsilon(x)$	approximation to g(x)
$U(x)$	unit step function
$\delta(x)$	delta function
I_ϵ	approximation to integral I
ν, B	weighting parameters; (32)
μ	dimensionality parameter of array; (38)
$w(x)$	required array weighting; (32), (37)
$w_\epsilon(x)$	approximate array weighting; (36), (53)
u	array parameter containing geometry, wavelength, and look and steering angles; (38)
$v(u)$	array voltage response; (38)
$\mathcal{J}_\alpha(z)$	Bessel function ratio; (40)
$v_i(u)$	ideal voltage pattern; (41)
$v_\epsilon(u)$	approximate voltage pattern; (59)
$n_\epsilon(x)$	neutralizer; figure 7
y	linear transformation of x; (54)

INTRODUCTION

Evaluation of integrals involving generalized functions is often accomplished via an integration by parts, without regard to an interpretation of the behavior of the function at its singular points. See, for example, ref. 1, eqs. (I-17), (I-18), and (I-32); thus, there follows an integral such as

$$\int_{-\infty}^{\infty} dx\, \frac{\cos(wx)}{x^2} = -\pi |w| \,. \qquad (1)$$

It is difficult to interpret and attach physical significance to this integral; in fact, the major contribution to the integral in (1) comes from the neighborhood of $x = 0$, where the integrand appears to be positive and not integrable, yet the right-hand side of (1) is negative and finite. We would like to approximate the generalized function $1/x^2$ in (1) and get a physical interpretation that is consistent with the result given by (1).

In a recent study of the ideal patterns for arrays in one, two, and three dimensions, it was found that the required weightings were impulsive or more singular than an impulse, depending on the dimensionality of the application; see ref. 2. In order to make these results of practical utility, it is necessary to approximate these singular weightings (generalized functions) by finite-valued functions and thereby realize approximations to the ideal patterns. This approximation procedure and its performance will be detailed here.

APPROXIMATION PROCEDURE

We take as given the possibility of approximating a delta function $\delta(x)$ by finite-valued functions; see, for example, ref. 1, page 279, where a one-sided rectangular pulse and a two-sided Gaussian pulse are used for illustration purposes. Also, on page 280, a one-sided approximation to the doublet $\delta'(x)$ is given. Alternative approximations are presented in ref. 3, pages 11-12.

Let us suppose that f(x) is an ordinary function which is integrable at $x = a+$, but that generalized function (ref. 3, page 30)

$$g(x) \equiv f'(x) \tag{2}$$

is not integrable at $x = a+$. For example, with $a = 0$,

$$f(x) = \begin{Bmatrix} -2x^{-1/2} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{Bmatrix}, \tag{3}$$

$$g(x) = \begin{Bmatrix} x^{-3/2} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{Bmatrix}, \tag{4}$$

is such a pair; see ref. 3, definition 8. Yet integration by parts yields

$$\int_{-\infty}^{\infty} dx\; g(x)\; h(x) = -\int_{-\infty}^{\infty} dx\; f(x)\; h'(x) = 2\int_{0}^{\infty} dx\; x^{-1/2}\, h'(x)\;, \tag{5}$$

which is integrable for any good function h(x).

To make sense of this situation, we approximate function f(x) by function $f_\varepsilon(x)$ as shown in figure 1; that is, $f_\varepsilon(x)$ has constant value $f(a+\varepsilon)$ in the neighborhood ε of a. The approximation to generalized function g(x) that we adopt is

$$g_\varepsilon(x) = f'_\varepsilon(x) = f(a+\varepsilon)\,\delta(x-a) + f'(x)\,U(x-a-\varepsilon)\ , \tag{6}$$

as depicted in figure 2, where U(x) is the unit step function; that is, $g_\varepsilon(x)$ has an impulse of area $f(a+\varepsilon)$ at $x = a$, and is equal to $f'(x)$ for $x > a+\varepsilon$. (We can approximate the impulse as discussed at the beginning of this section.)

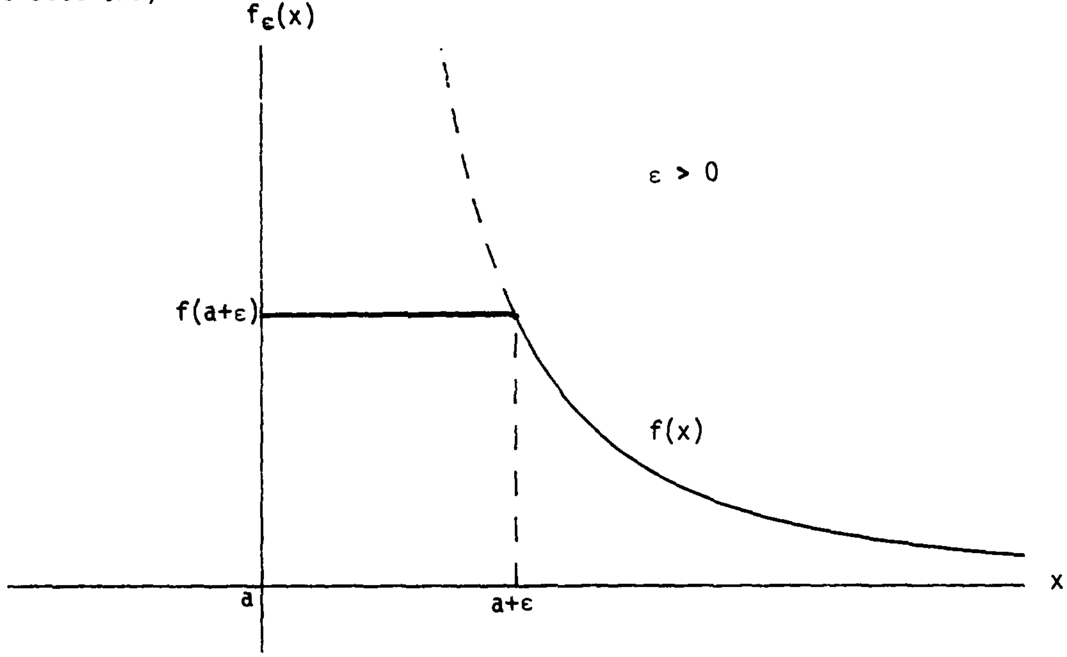

Figure 1. Approximation $f_\varepsilon(x)$ to $f(x)$.

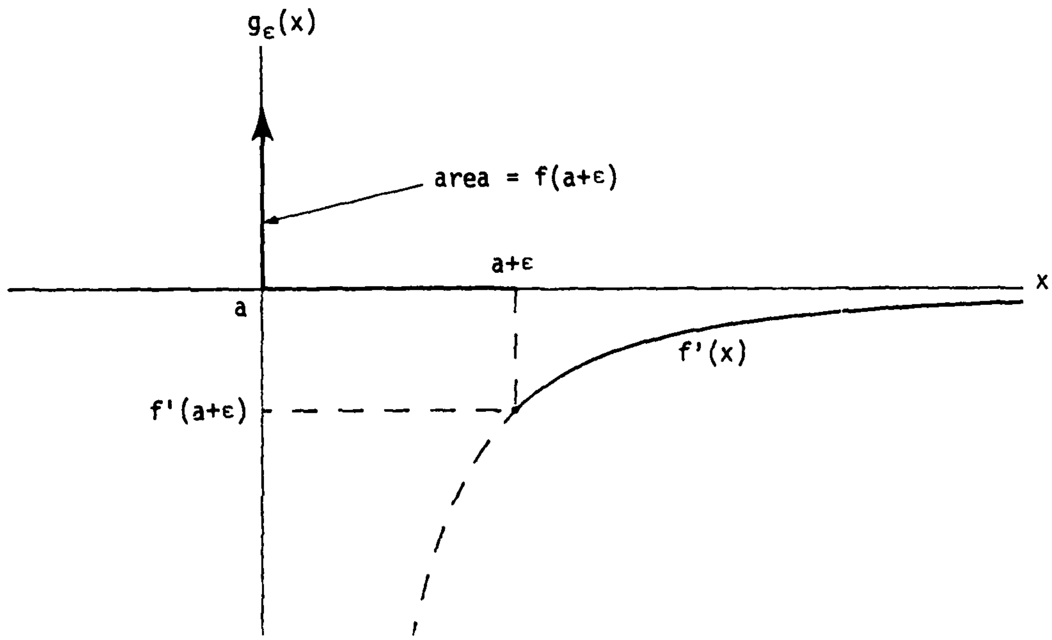

Figure 2. Approximation $g_\varepsilon(x)$ to $g(x)$.

It is important to observe that if $f(x)$ is singular (ref. 3, definition 21) at $x = a$, as for example (3), the area $f(a+\varepsilon)$ of the impulse at $x = a$ becomes progressively larger as $\varepsilon \to 0$. However, the area under the remaining portion of $g_\varepsilon(x)$ also becomes larger, but in such a fashion that the total area under $g_\varepsilon(x)$ remains finite as $\varepsilon \to 0$; this will be shown below in (11).

To see the effect of this approximation on an integral of a generalized function $g(x)$, consider integral

$$I \equiv \int_a^b dx\; g(x)\; h(x) \; . \tag{7}$$

The approximation to I is:

$$I_\varepsilon \equiv \int_a^b dx\; g_\varepsilon(x)\; h(x)$$

$$= f(a+\varepsilon)\; h(a) + \int_{a+\varepsilon}^b dx\; f'(x)\; h(x) \; , \tag{8}$$

where we employed (6). Integration by parts on (8) yields two alternative forms:

$$I_\varepsilon = f(a+\varepsilon)\Big[h(a) - h(a+\varepsilon)\Big] + f(b)\; h(b) - \int_{a+\varepsilon}^b dx\; f(x)\; h'(x) \tag{9A}$$

$$= f(a+\varepsilon)\Big[h(a) - h(a+\varepsilon)\Big] + \int_a^{a+\varepsilon} dx\; f(x)\; h'(x)$$

$$+ f(b)\; h(b) - \int_a^b dx\; f(x)\; h'(x) \; . \tag{9B}$$

Since $f(x)$ is integrable at $x = a+$, the leading term in (9A) and the two leading terms in (9B) both approach zero as $\varepsilon \to 0$, yielding the limit

$$I = \lim_{\varepsilon \to 0} I_\varepsilon = f(b)\, h(b) - \int_a^b dx\, f(x)\, h'(x) \; . \tag{10}$$

This is the value of (7) expressed in terms of integrable functions.

The area under approximation $g_\varepsilon(x)$, for $\varepsilon > 0$, is available by substituting $h(x) = 1$ in (8) and (9):

$$\int_a^b dx\, g_\varepsilon(x) = f(b) \; , \tag{11}$$

which is finite and independent of ε. Thus, the impulse in figure 2 is necessary in order to compensate for the increasing area that develops under $f'(x)$ near $x = a$, when ε approaches zero.

SINGULARITY AT $x = b-$

Suppose, instead, that $f(x)$ is integrable at $x = b-$, but that generalized function (2) is not. By employing a procedure similar to that above, we have approximation

$$g_\varepsilon(x) = -f(b-\varepsilon)\, \delta(x-b) + f'(x)\, U(b-\varepsilon-x) \tag{12}$$

to $g(x)$. The integral I_ε can then be expressed as

$$I_\varepsilon = \int_a^b dx\ g_\varepsilon(x)\ h(x)$$

$$= -f(b-\varepsilon)\ h(b) + \int_a^{b-\varepsilon} dx\ f'(x)\ h(x)$$

$$= -f(b-\varepsilon)\left[h(b) - h(b-\varepsilon)\right] - f(a)\ h(a) - \int_a^{b-\varepsilon} dx\ f(x)\ h'(x)\ . \qquad (13)$$

There follows

$$\int_a^b dx\ g(x)\ h(x) = I = \lim_{\varepsilon \to 0} I_\varepsilon = -f(a)\ h(a) - \int_a^b dx\ f(x)\ h'(x)\ , \qquad (14)$$

in terms of integrable functions.

The area under approximation $g_\varepsilon(x)$ is finite and independent of ε: let $h(x) = 1$ in (13) and get directly

$$\int_a^b dx\ g_\varepsilon(x) = -f(a)\ . \qquad (15)$$

RELATION TO "FINITE PART" OF INTEGRAL

If we start with integral (7) and integrate by part , there follows

$$I = \Big[f(x)\ h(x)\Big]_a^b - \int_a^b dx\ f(x)\ h'(x)\ . \qquad (16)$$

Now if $f(x)$ is singular at $x = a$, the finite part (ref. 3, page 32) of (16) is obtained by dropping the term involving $f(a)$, thereby obtaining identically (10). Conversely, if $f(x)$ is singular at $x = b$, the finite part of (16) is just (14). Thus, the limit of the approximation procedure developed here is exactly what is yielded by the finite part procedure.

EXAMPLES

EXAMPLE 1

$$f(x) = \frac{x^{\nu+1}}{\nu+1} \quad \text{for } x > 0 \quad \text{where } -2 < \nu < -1 \; . \tag{17}$$

This function is singular but integrable at $x = 0$. In figure 1, we identify

$$a = 0 \tag{18}$$

and get

$$f'(x) = x^{\nu} \quad \text{for } x > 0 \; . \tag{19}$$

Then, from (6), the approximation to the generalized function $x^{\nu}\, U(x)$ is

$$g_{\varepsilon}(x) = \frac{\varepsilon^{\nu+1}}{\nu+1}\, \delta(x) + x^{\nu}\, U(x-\varepsilon) \tag{20}$$

and is depicted in figure 3.

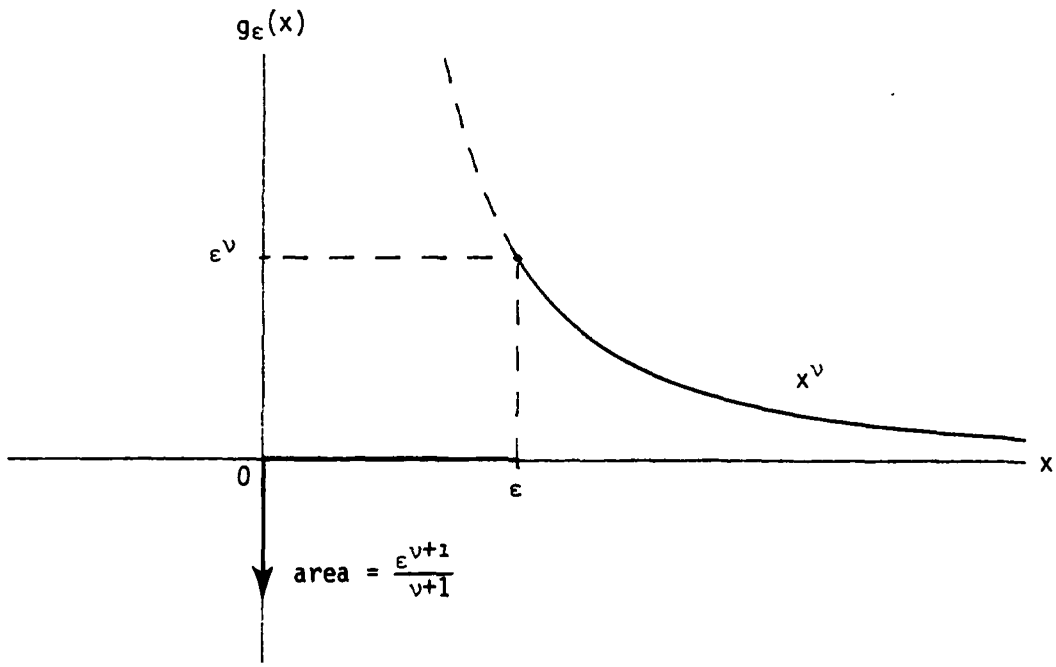

Figure 3. Approximation to Generalized Function $x^{\nu}\, U(x)$.

The result of applying (20) to a good function h(x) is available from (8) and (17),

$$\int_0^b dx\; g_\epsilon(x)\; h(x) = \frac{\epsilon^{\nu+1}}{\nu+1}\, h(0) + \int_\epsilon^b dx\; x^\nu\; h(x) \;, \tag{21}$$

or from either form in (9). The limit follows from (10) and (7):

$$I = \frac{b^{\nu+1}}{\nu+1}\, h(b) - \int_0^b dx\; \frac{x^{\nu+1}}{\nu+1}\, h'(x) = \int_0^b dx\; x^\nu\; h(x) \;. \tag{22}$$

The following examples are derived in similar fashion; just the results are listed.

EXAMPLE 2

$$f(x) = \ln x \text{ for } x > 0$$

$$a = 0$$

$$f'(x) = \frac{1}{x} \quad \text{for } x > 0$$

$$g_\epsilon(x) = \ln \epsilon\; \delta(x) + \frac{1}{x}\, U(x-\epsilon)$$

$$I = \ln b\; h(b) - \int_0^b dx\; \ln x\; h'(x) = \int_0^b dx\; \frac{1}{x}\, h(x) \;. \tag{23}$$

EXAMPLE 3

$$f(x) = \tfrac{1}{2}(\ln x)^2 \quad \text{for } x > 0$$

$$a = 0$$

$$f'(x) = \frac{\ln x}{x} \qquad \text{for } x > 0$$

$$g_\varepsilon(x) = \tfrac{1}{2}(\ln \varepsilon)^2\, \delta(x) + \frac{\ln x}{x}\, U(x-\varepsilon)$$

$$I = \tfrac{1}{2}(\ln b)^2\, h(b) - \int_0^b dx\, \tfrac{1}{2}(\ln x)^2\, h'(x) = \int_0^b dx\, \frac{\ln x}{x}\, h(x)\ . \quad (24)$$

EXAMPLE 4

$$f(x) = \frac{x^{\nu+1}}{\nu+1}\left[\ln x - \frac{1}{\nu+1}\right] \quad \text{for } x > 0\ ;\ \nu > -2$$

$$a = 0$$

$$f'(x) = x^\nu \ln x \text{ for } x > 0$$

$$g_\varepsilon(x) = \frac{\varepsilon^{\nu+1}}{\nu+1}\left[\ln \varepsilon - \frac{1}{\nu+1}\right] \delta(x) + x^\nu \ln x\, U(x-\varepsilon)$$

$$I = \frac{b^{\nu+1}}{\nu+1}\left[\ln b - \frac{1}{\nu+1}\right] h(b) - \int_0^b dx\, \frac{x^{\nu+1}}{\nu+1}\left[\ln x - \frac{1}{\nu+1}\right] h'(x)$$

$$= \int_0^b dx\, x^\nu \ln x\, h(x)\ . \quad (25)$$

EXAMPLE 5

Here we consider the result for the generalized function $1/x^2$ as presented in the integral (1). Consider the ordinary function

$$f_\varepsilon(x) \equiv \left\{\begin{array}{ll} -1/x & \text{for } |x| > \varepsilon \\ -\frac{1}{\varepsilon}\,\mathrm{sgn}(x) & \text{for } |x| < \varepsilon \end{array}\right\} , \tag{26}$$

and its derivative

$$g_\varepsilon(x) = f'_\varepsilon(x) = \left\{\begin{array}{ll} 1/x^2 & \text{for } |x| > \varepsilon \\ -\frac{2}{\varepsilon}\,\delta(x) & \text{for } |x| < \varepsilon \end{array}\right\} ; \tag{27}$$

see figure 4. The function $g_\varepsilon(x)$ is an approximation to generalized function

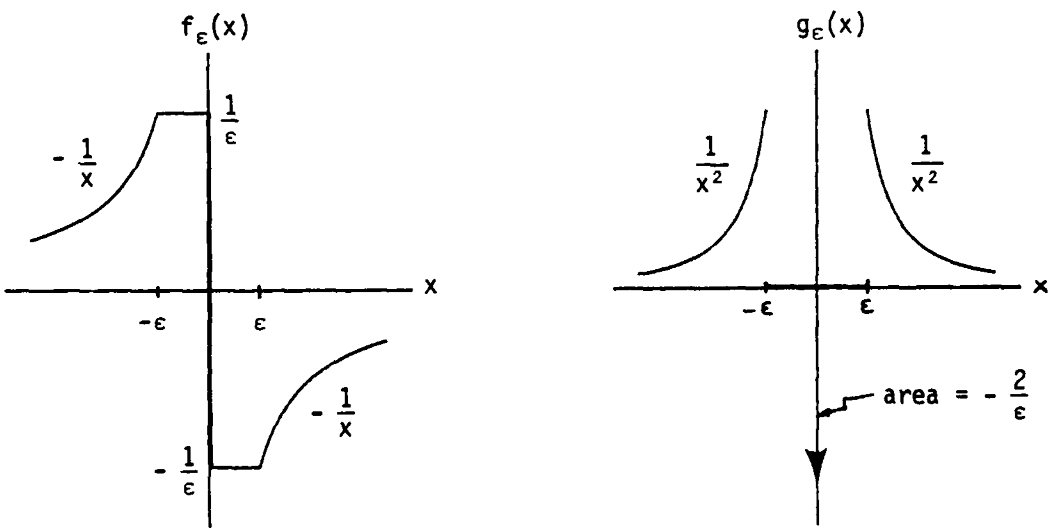

Figure 4. Approximation to Generalized Function $\frac{1}{x^2}$

$1/x^2$; it contains an impulse of area $-2/\epsilon$ (at $x = 0$) that tends to $-\infty$ as $\epsilon \to 0$.

The approximation to integral (7) is as defined in (8); namely, for $a = -\infty$, $b = +\infty$,

$$I_\epsilon = \int_{-\infty}^{+\infty} dx\; g_\epsilon(x)\cos(wx) = 2\int_{\epsilon}^{+\infty} dx\,\frac{\cos(wx)}{x^2} - \frac{2}{\epsilon}$$

$$= -\pi|w| - 2\,\frac{1-\cos(w\epsilon)}{\epsilon} + 2w\,\mathrm{Si}(w\epsilon), \tag{28}$$

where Si is the sine integral (ref. 4, eq. 5.2.1). The limit as $\epsilon \to 0$ gives the result for the generalized function $1/x^2$, namely,

$$\int_{-\infty}^{+\infty} dx\,\frac{1}{x^2}\cos(wx) = -\pi|w|. \tag{29}$$

This result is in agreement with ref. 1, eq. I-32, and with ref. 3, page 43, for x^{-m} with $m = 2$.

An alternative approximation to generalized function $1/x^2$ that uses finite functions is

$$g_\epsilon(x) = \left\{\begin{matrix} 1/x^2 & \text{for } |x| > \epsilon \\ 0 & \text{for } |x| < \epsilon \end{matrix}\right\} - \frac{2}{\epsilon}\frac{1}{\sqrt{2\pi\epsilon}}\exp\left(-\frac{x^2}{2\epsilon^2}\right). \tag{30}$$

There follows

$$I_\epsilon = \int_{-\infty}^{+\infty} dx\; g_\epsilon(x)\, \cos(wx)$$

$$= -\pi|w| + \frac{2}{\epsilon}\left[\cos(w\epsilon) - \exp\left(-\frac{1}{2}w^2\epsilon^2\right)\right] + 2w\,\mathrm{Si}(w\epsilon) \quad . \qquad (31)$$

The limit as $\epsilon \rightarrow 0$ is again exactly (29). The essential feature of approximations (27) and (30) is the large, sharp negative pulse of area $-2/\epsilon$ that develops about $x = 0$ as ϵ gets small.

APPLICATION TO ARRAY WEIGHTING

This example relies heavily on material presented in ref. 2 on array weighting for good voltage response patterns in one, two, and three dimensions. In particular, from ref. 2, eq. E-7: for $-2 < \nu < -1$, we have, for the required array weighting $w(x)$ to realize pattern $\mathcal{J}_{\mu+\nu+1}\left(\sqrt{u^2-B^2}\right)$, in one, two, or three dimensions $\left(\mu = -\frac{1}{2},\ 0,\ \text{or}\ \frac{1}{2}\right)$, the relation

$$-x\ w(x) = \frac{d}{dx}\left\{\left(\frac{\sqrt{1-x^2}}{B}\right)^{\nu+1} I_{\nu+1}\left(B\sqrt{1-x^2}\right)U(1-x)\right\} \quad \text{for } 0 < x\ , \tag{32}$$

and zero otherwise. To match (2), we identify

$$g(x) = -x\ w(x)\ , \tag{33}$$

and

$$f(x) = \left(\frac{\sqrt{1-x^2}}{B}\right)^{\nu+1} I_{\nu+1}\left(B\sqrt{1-x^2}\right)U(1-x) \text{ for } 0 < x\ . \tag{34}$$

The function $f(x)$ is singular but integrable at $x = 1-$, since $-1 < \nu+1 < 0$; thus the derivative in (32) will generate a generalized function (ref. 3, page 30) for the weighting $w(x)$.

We now identify $b = 1$ in (12) and obtain, via (34) and ref. 4, eq. 9.6.28, the approximation

$$\begin{aligned} g_\epsilon(x) = &-\left(\frac{\sqrt{2\epsilon-\epsilon^2}}{B}\right)^{\nu+1} I_{\nu+1}\left(B\sqrt{2\epsilon-\epsilon^2}\right)\ \delta(x-1) \\ &-x\left(\frac{\sqrt{1-x^2}}{B}\right)^{\nu} I_{\nu}\left(B\sqrt{1-x^2}\right)\ U(1-\epsilon-x) \quad \text{for } 0 < x\ . \end{aligned} \tag{35}$$

Then there follows from (33), the approximate weighting

$$w_\epsilon(x) = \left(\frac{\sqrt{2\epsilon-\epsilon^2}}{B}\right)^{\nu+1} I_{\nu+1}\left(B\sqrt{2\epsilon-\epsilon^2}\right) \delta(x-1)$$

$$+ \left(\frac{\sqrt{1-x^2}}{B}\right)^{\nu} I_\nu\left(B\sqrt{1-x^2}\right) U(1-\epsilon-x) \quad \text{for } 0 < x \ . \tag{36}$$

This is the result presented in ref. 2, eq. F-14. It is an approximation to the generalized function

$$w(x) = \left(\frac{\sqrt{1-x^2}}{B}\right)^{\nu} I_\nu\left(B\sqrt{1-x^2}\right) U(1-x)\ U(x) \ , \tag{37}$$

which is the required array weighting according to ref. 2, eq. E-7.

We will carry this example further than the previous ones, by determining the voltage response pattern that is actually realized by an array employing approximation weighting (36) rather than (37). The array voltage response pattern is given by ref. 2, eqs. 1-12, for any μ, as

$$v(u) = \int_0^1 dx\ x\left(\frac{x}{u}\right)^{\mu} J_\mu(ux)\ w(x) \ , \tag{38}$$

where the parameter μ determines the array dimensionality, and u contains the array geometry, the plane-wave arrival wavelength, and the various look and steering angles. Substitution of generalized function (37) in (38) yields pattern (ref. 2, eq. E-1)

$$v(u) = \mathcal{J}_{\mu+\nu+1}\left(\sqrt{u^2-B^2}\right) \quad \text{for all } u \ , \tag{39}$$

where we define entire function (ref. 2, eqs. 17-21)

$$\mathcal{J}_\alpha(z) \equiv \frac{J_\alpha(z)}{z^\alpha} \ . \tag{40}$$

The particular case of $\mu+\nu = -1.5$ in (39) yields ideal voltage pattern

$$v_i(u) = \mathcal{J}_{-\frac{1}{2}}\left(\sqrt{u^2-B^2}\right) = \left(\frac{2}{\pi}\right)^{1/2}\cos\left(\sqrt{u^2-B^2}\right) = \left(\frac{2}{\pi}\right)^{1/2}\cosh\left(\sqrt{B^2-u^2}\right) . \quad (41)$$

This relation is true for all u, whether larger or smaller than B. It indicates a mainlobe at $u = 0$ of amplitude $\left(\frac{2}{\pi}\right)^{1/2}\cosh(B)$, and sidelobes for $u > B$, all of equal amplitude $\pm\left(\frac{2}{\pi}\right)^{1/2}$. Plots of this ideal pattern are available in ref. 2, figures 13-16.

When approximation $w_\varepsilon(x)$ in (36) is substituted, instead, in (38), we get several equivalent expressions for the corresponding pattern $v_\varepsilon(u)$, as given in appendix A. Plots of typical results are presented in figures 5 and 6 for $\nu = -1.5$. In both figures, the curve labeled $\varepsilon = 0$ is the desired pattern (39). Figure 5 corresponds to a volumetric-spherical array ($\mu = .5$), and the desired pattern is, from (39),

$$\mathcal{J}_o\left(\sqrt{u^2-B^2}\right) = J_o\left(\sqrt{u^2-B^2}\right) = I_o\left(\sqrt{B^2-u^2}\right), \quad (42)$$

which decays at a 3 dB/octave rate for large u. Figure 6 corresponds to a planar-circular array ($\mu = 0$) with desired pattern equal to ideal pattern (41). The approximations in both of these figures for $\varepsilon = .1$ are quite good, but those for $\varepsilon = .2$ have undergone significant degradation. The possibility of replacing the delta function in (36) by a narrow pulse is considered in the next section.

Figure 5 furnishes an approximation to the bottom asterisked case in ref. 2, figure 12; figure 6 does the same for the middle asterisked case in ref. 2, figure 13. The three asterisked cases for $\nu = -1$ in ref. 2, figures 11-13, merely require delta functions at $x = 1$ and are considered solved. The last remaining asterisked case is the bottom one in ref. 2, figure 13, for $\nu = -2$. But this has already been shown in ref. 2, eqs. E-37 - E-39, to involve a delta function and its derivative, both of which are easily approximated; see, for example, ref. 1, pages 279-280, or ref. 2, pages 11-12.

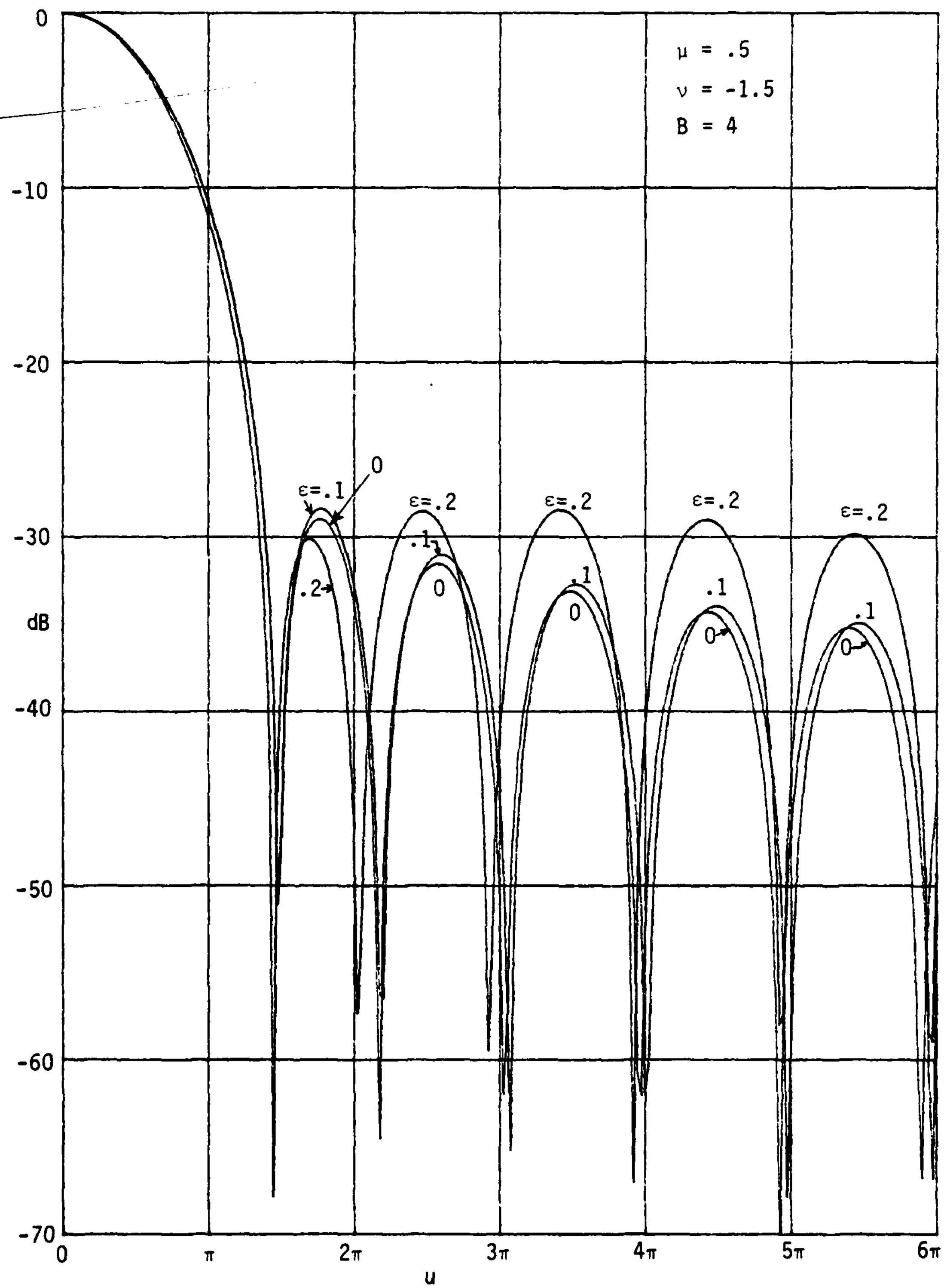

Figure 5. Pattern of Spherical Array for Weighting (36)

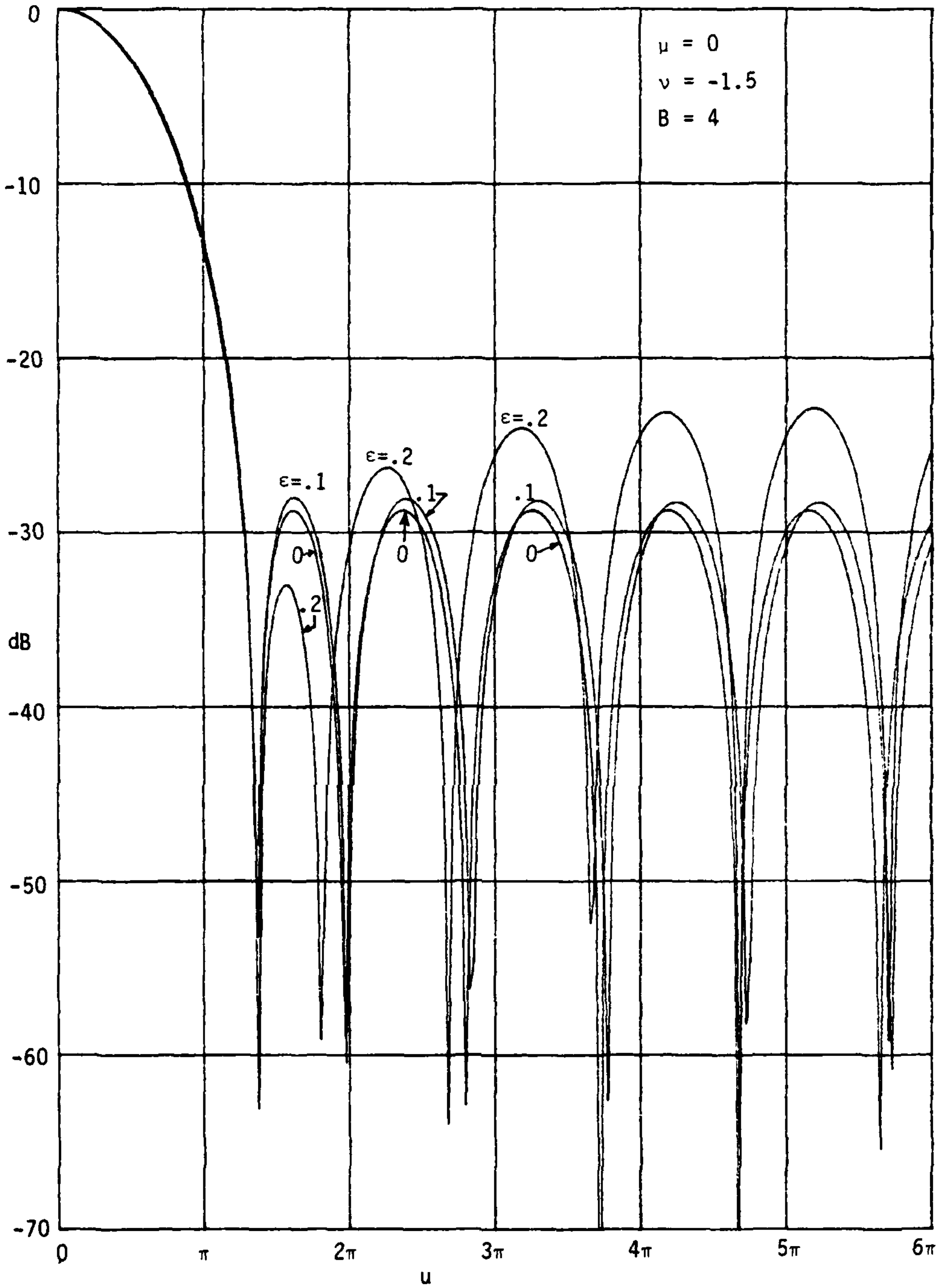

Figure 6. Pattern of Circular Array for Weighting (36)

USE OF NEUTRALIZERS

An alternative smoother approximation to f(x) and g(x) than afforded by figures 1 and 2 is by the use of neutralizers; see ref. 5, section 3.3. Consider the neutralizer $n_\varepsilon(x)$ and its derivative shown in figure 7; the neutralizer is 0 at x = a, and 1 at x = b. Furthermore, it has derivatives

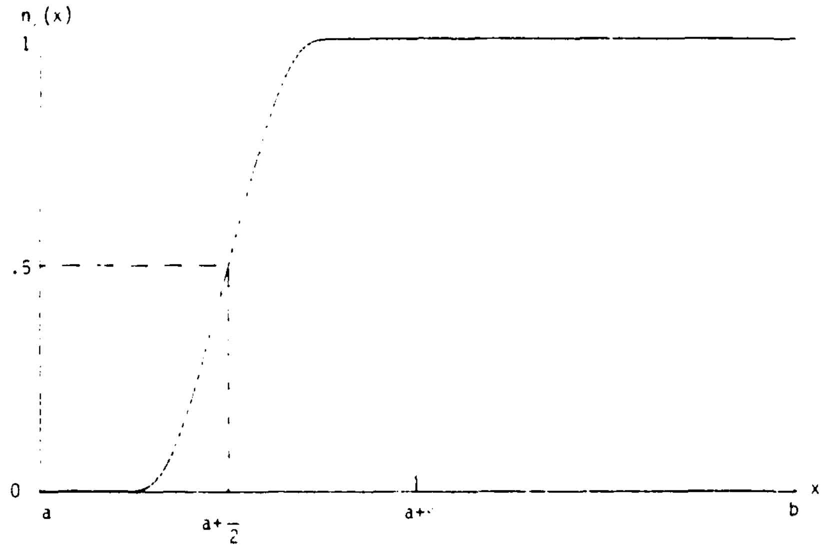

Figure 7A. Function $n_\varepsilon(x)$

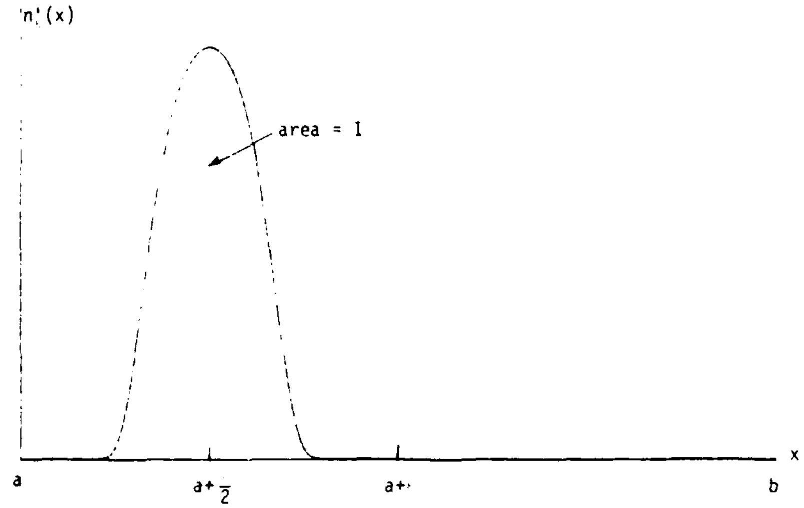

Figure 7B. Function $n_\varepsilon'(x)$

Figure 7. Neutralizer $n_\varepsilon(x)$ and its Derivative

of all orders, all of which are zero at end points a and b. The positive parameter ϵ characterizes the critical point, $x_c = a+\frac{\epsilon}{2}$, where the neutralizer is 1/2; this point x_c will approach a as $\epsilon \to 0+$. The neutralizer has completed its transition to 1 by the value $x = a+\epsilon$. $n_\epsilon(x)$ will approach 1 for all $x > a$, as $\epsilon \to 0$.

The approximation we take to f(x) is the product

$$f_\epsilon(x) \equiv f(x)\, n_\epsilon(x) \ , \tag{43}$$

and the corresponding approximation to generalized function (2) is the smooth function

$$g_\epsilon(x) = f_\epsilon{}'(x) = f'(x)\, n_\epsilon(x) + f(x)\, n_\epsilon{}'(x) \ . \tag{44}$$

This approach is similar to the regular sequence of good functions used to define a generalized function in ref. 3, pages 16-17. A representative example is depicted in figure 8. As before, the area under approximation $g_\epsilon(x)$ is independent of ϵ:

$$\int_a^b dx\ g_\epsilon(x) = \Big[f_\epsilon(x)\Big]_a^b = f_\epsilon(b) = f(b) \ . \tag{45}$$

The result of applying $g_\epsilon(x)$ to function h(x) is now

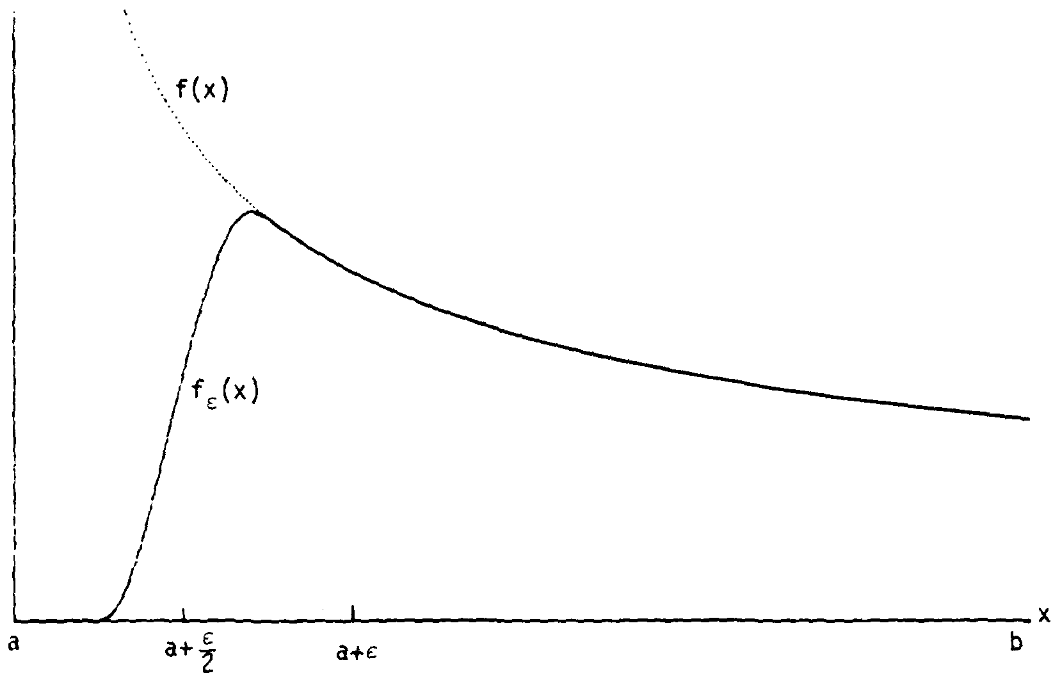

Figure 8A. Function $f_\varepsilon(x)$

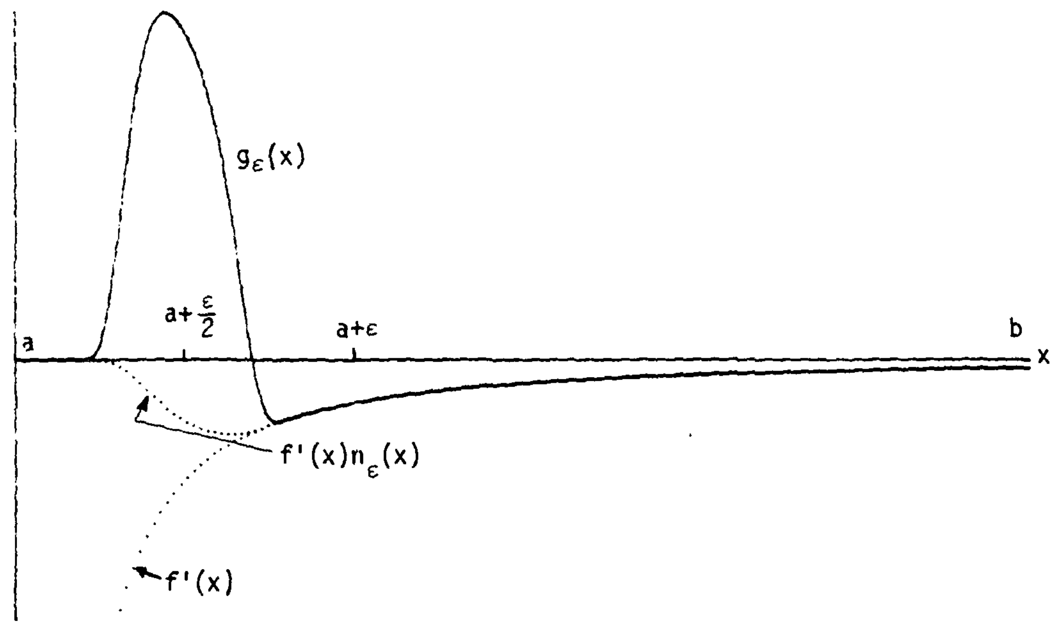

Figure 8B. Function $g_\varepsilon(x)$

Figure 8. Approximation to Generalized Function via Neutralizer

$$I_\varepsilon = \int_a^b dx\; g_\varepsilon(x)\; h(x) = \int_a^b dx\; f_\varepsilon{}'(x)\; h(x)$$

$$= \Big[f_\varepsilon(x)\; h(x)\Big]_a^b - \int_a^b dx\; f_\varepsilon(x)\; h'(x)$$

$$= f(b)\; h(b) - \int_a^b dx\; f(x)\; n_\varepsilon(x)\; h'(x)\ , \tag{46}$$

where we have utilized the properties of the neutralizer. Equation (46) now replaces (8). In the limit of $\varepsilon \to 0$, (46) yields

$$I = \lim_{\varepsilon \to 0} I_\varepsilon = f(b)\; h(b) - \int_a^b dx\; f(x)\; h'(x)\ , \tag{47}$$

in agreement with (10).

A limiting case of figure 7 is a step-function at $x = a+\frac{\varepsilon}{2}$; i.e.,

$$n_\varepsilon(x) = U(x-a-\tfrac{\varepsilon}{2})\ . \tag{48}$$

Then

$$n_\varepsilon{}'(x) = \delta(x-a-\tfrac{\varepsilon}{2}) \tag{49}$$

and (44) yields

$$g_\varepsilon(x) = f'(x)\; U(x-a-\tfrac{\varepsilon}{2}) + f(a+\tfrac{\varepsilon}{2})\; \delta(x-a-\tfrac{\varepsilon}{2})\ . \tag{50}$$

This is almost identical to (6), which has its impulse located at $x = a$ rather than at $x = a+\frac{\varepsilon}{2}$.

APPLICATION TO ARRAY

The starting point is (32), again, for the required weighting. The function f(x) is identified as in (34), and we get, from (33) and (44),

$$-xw_\varepsilon(x) = g_\varepsilon(x) = f'(x)\, n_\varepsilon(x) + f(x)\, n_\varepsilon'(x). \tag{51}$$

There follows, from (34) and ref. 4, eq. 9.6.28,

$$f'(x) = -x\left(\frac{\sqrt{1-x^2}}{B}\right)^{\nu} I_\nu\left(B\sqrt{1-x^2}\right) \quad \text{for } 0 < x < 1 . \tag{52}$$

Substitution of (34) and (52) in (51) yields, for the approximate weighting,

$$w_\varepsilon(x) = \left(\frac{\sqrt{1-x^2}}{B}\right)^{\nu} I_\nu\left(B\sqrt{1-x^2}\right) n_\varepsilon(x)$$
$$- \frac{1}{x}\left(\frac{\sqrt{1-x^2}}{B}\right)^{\nu+1} I_{\nu+1}\left(B\sqrt{1-x^2}\right) n_\varepsilon'(x) \quad \text{for } 0 < x < 1 . \tag{53}$$

However, the neutralizer in this case must be chosen to be 1 at $x = 0$, and 0 at $x = 1$; that is, it is a reflected version of figure 7A. For our purposes, it is not necessary for the neutralizer to have derivatives of all orders. Rather, we select $n_\varepsilon(x)$ so that $n_\varepsilon''(x)$ is continuous for all x, and such that $n_\varepsilon'(x)$ and $n_\varepsilon''(x)$ are zero at the edges of the transition region $(1-\varepsilon, 1)$; see figure 9. Here, letting

$$y = \frac{x-1+\frac{\varepsilon}{2}}{\varepsilon}, \tag{54}$$

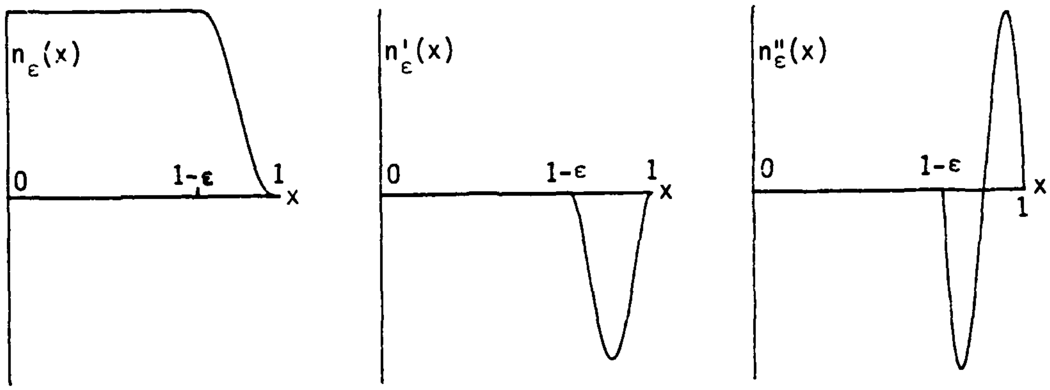

Figure 9. Neutralizer (55) and its Derivatives

we take

$$n_\varepsilon(x) = \frac{1}{2} - \frac{1}{8}\left(15y - 40y^3 + 48y^5\right) \quad \text{for } 1-\varepsilon \le x \le 1, \qquad (55)$$

and 1 for $0 \le x \le 1-\varepsilon$. Then

$$n'_\varepsilon(x) = -\frac{15}{8\varepsilon}\left(1-8y^2+16y^4\right) \quad \text{for } 1-\varepsilon \le x \le 1, \qquad (56)$$

and zero otherwise. Also,

$$n''_\varepsilon(x) = \frac{30}{\varepsilon^2}\left(y-4y^3\right) \text{ for } 1-\varepsilon \le x \le 1, \qquad (57)$$

and zero otherwise. From (55)-(57), there follows

$$\left.\begin{aligned} n_\varepsilon(x) &\sim \frac{10}{\varepsilon^3}\left(1-x\right)^3 \\ n_\varepsilon'(x) &\sim -\frac{30}{\varepsilon^3}\left(1-x\right)^2 \\ n_\varepsilon''(x) &\sim \frac{60}{\varepsilon^3}\left(1-x\right) \end{aligned}\right\} \quad \text{as } x \to 1- . \tag{58}$$

Combining this with the behavior of the Bessel function near zero argument (ref. 4, eq. 9.6.7), we find that both terms of the approximate weighting $w_\varepsilon(x)$ in (53) are proportional to $(1-x)^{3+\nu}$ as x approaches 1. Thus if $\nu \geq -2$, the approximate weighting will approach zero at least linearly at x = 1. An example of $w_\varepsilon(x)$ for $\nu = -1.5$ and B = 4 is given in figure 10

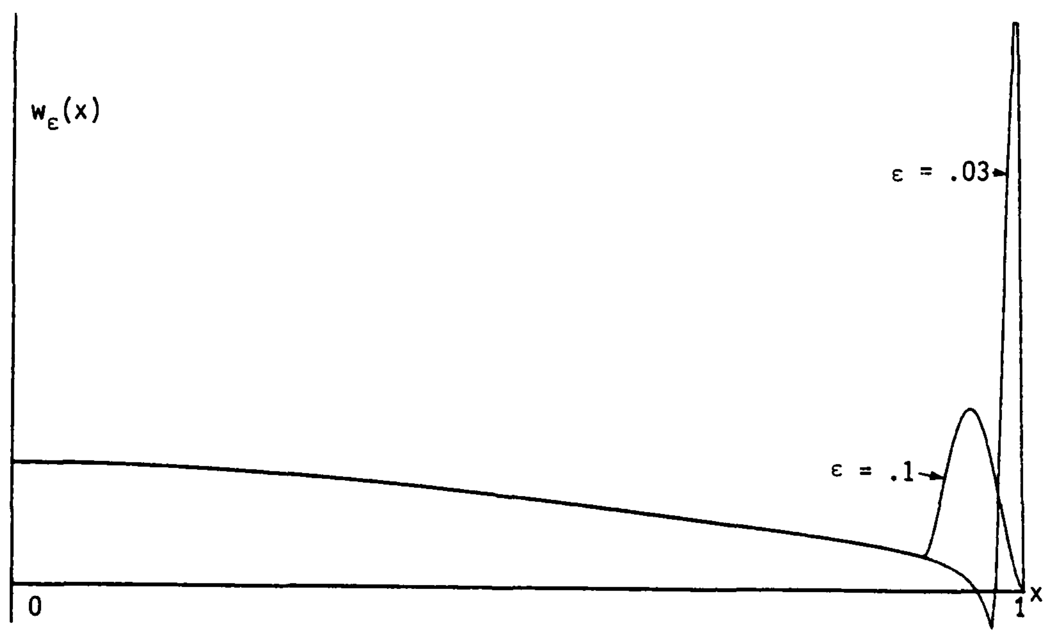

Figure 10. Approximate Weighting $w_\varepsilon(x)$ for $\nu = -1.5$, B = 4

for two different selections of ϵ. The large positive spike near $x = 1$ is very pronounced for small ϵ.

The array voltage response corresponding to approximate weighting $w_\epsilon(x)$ in (53) is given by (38) as

$$v_\epsilon(u) = \int_0^1 dx\, x \left(\frac{x}{u}\right)^\mu J_\mu(ux)\, w_\epsilon(x)$$

$$= \int_0^1 dx\, x^{2\mu+1}\, \mathcal{J}_\mu(ux)\, w_\epsilon(x) \; , \qquad (59)$$

where we used (40). A program for the evaluation of (59) (along with (53)-(56)) is presented in appendix B. Sample responses are plotted in figure 11 for a spherical array and in figure 12 for a circular array. Comparison with the corresponding results in figures 5 and 6 reveals that the impulsive weighting of (36) yields a better approximation to the ideal pattern ($\epsilon = 0$) than the smoother weighting of (53). That is, ϵ must be chosen smaller in (53) than in (36), in order to realize approximately the same voltage response pattern in the first few sidelobes.

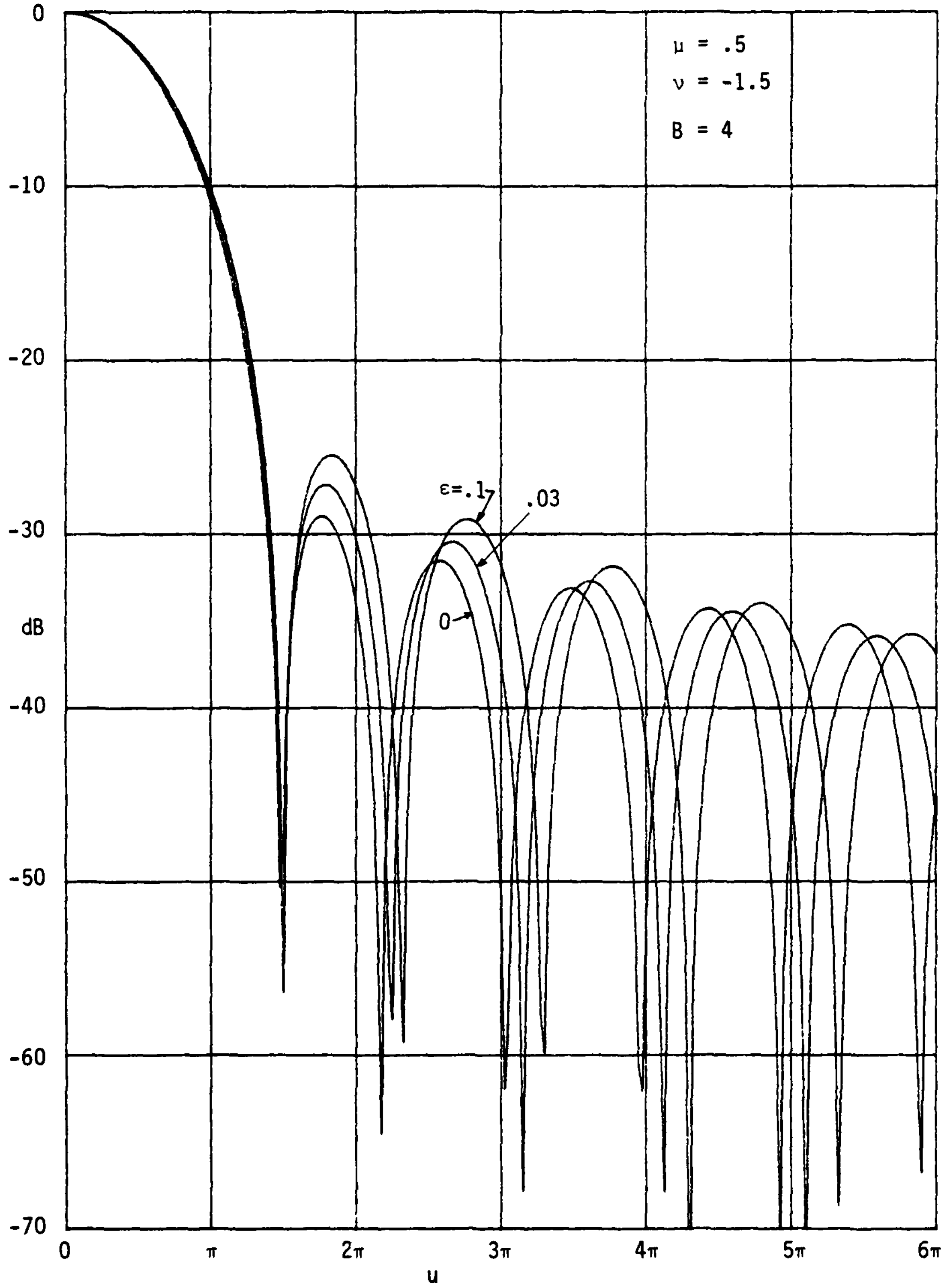

Figure 11. Pattern of Spherical Array for Weighting (53)

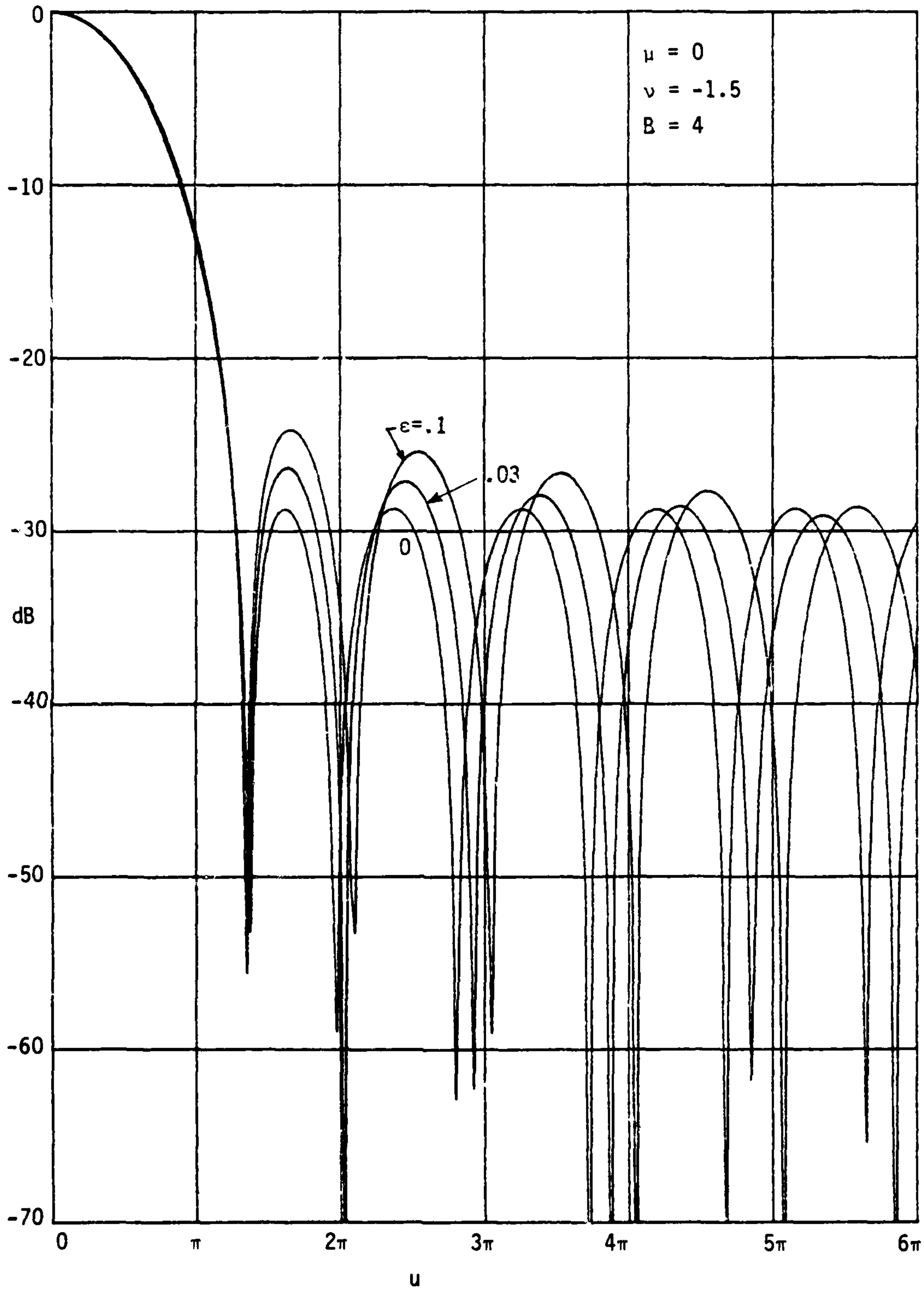

Figure 12. Pattern of Circular Array for Weighting (53)

SUMMARY

We have indicated how approximations to some generalized functions generated by the process of differentiation may be realized, and then have applied the procedure to the approximation of the weighting required to realize the ideal response patterns of arrays in several dimensions. Two examples, one impulsive and the other smooth, were used for circular and spherical arrays, and the resultant approximate patterns were plotted for different choices of the parameter ϵ governing the transition region near the singularity of the generalized function. How small ϵ must be chosen depends on the form of the approximate weighting and the desired closeness to the specified pattern.

Appendix A
VOLTAGE RESPONSE PATTERN FOR WEIGHTING (36)

With the shorthand notation

$$f_\nu(x) \equiv \left(\frac{\sqrt{1-x^2}}{B}\right)^\nu I_\nu\left(B\sqrt{1-x^2}\right) \quad \text{for } 0 < x < 1 , \tag{A-1}$$

approximate weighting (36) takes the form

$$w_\epsilon(x) = f_{\nu+1}(1-\epsilon)\,\delta(x-1) + f_\nu(x)\,U(1-\epsilon-x) \quad \text{for } 0 < x . \tag{A-2}$$

Substitution of (A-2) in (38) yields, upon use of (40), the corresponding voltage response

$$v_\epsilon(u) = f_{\nu+1}(1-\epsilon)\,\mathcal{J}_\mu(u) + \int_0^{1-\epsilon} dx\; x^{2\mu+1}\,\mathcal{J}_\mu(ux)\,f_\nu(x) \tag{A-3}$$

for any μ; this is the relation programmed in ref. 2, appendix F.

For small ϵ (the case of most interest), the cusp of $f_\nu(x)$ in (A-3) at $x = 1$ causes numerical integration problems. We can alleviate this problem by integrating by parts on (A-3), using

$$\mathcal{U} = x^{2\mu}\,\mathcal{J}_\mu(ux), \quad \mathcal{V} = -\,f_{\nu+1}(x) , \tag{A-4}$$

to get

$$v_\epsilon(u) = f_{\nu+1}(1-\epsilon)\left[\mathcal{J}_\mu(u) - (1-\epsilon)^{2\mu}\,\mathcal{J}_\mu(u(1-\epsilon))\right] + \int_0^{1-\epsilon} dx\; x^{2\mu-1}\,\mathcal{J}_{\mu-1}(ux)\,f_{\nu+1}(x) \quad \text{for } \mu > 0 . \tag{A-5}$$

The function $f_{\nu+1}(x)$ is integrable at $x = 1$ if $\nu > -2$.

The completed integral over (0,1) in (A-5) can be evaluated in closed form (ref. 2), leading to

$$v_\epsilon(u) = \mathcal{J}_{\mu+\nu+1}\left(\sqrt{u^2-B^2}\right)$$

$$+ f_{\nu+1}(1-\epsilon)\left[\mathcal{J}_\mu(u) - (1-\epsilon)^{2\mu}\mathcal{J}_\mu(u(1-\epsilon))\right]$$

$$- \int_{1-\epsilon}^{1} dx\ x^{2\mu-1}\mathcal{J}_{\mu-1}(ux)\ f_{\nu+1}(x)\ , \qquad \text{(A-6)}$$

which is advantageous because the interval $(1-\epsilon,1)$ is small for small ϵ. Finally, the cusp at $x = 1$ can be eliminated for $\nu \geq -3/2$, by changing the variable of integration according to $x = \sin t$, getting

$$v_\epsilon(u) = \mathcal{J}_{\mu+\nu+1}\left(\sqrt{u^2-B^2}\right)$$

$$+ f_{\nu+1}(1-\epsilon)\left[\mathcal{J}_\mu(u) - (1-\epsilon)^{2\mu}\mathcal{J}_\mu(u(1-\epsilon))\right]$$

$$- \int_{A}^{\pi/2} dt\ (\sin t)^{2\mu-1}\ (\cos t)^{2\nu+3}\ \mathcal{J}_{\mu-1}(u \sin t)\ \mathcal{I}_{\nu+1}\ (B \cos t)\ , \qquad \text{(A-7)}$$

where $A = \text{arc}\sin(1-\epsilon)$, and

$$\mathcal{I}_\alpha(z) \equiv \frac{I_\alpha(z)}{z^\alpha}\ . \qquad \text{(A-8)}$$

Form (A-7) is good numerically for small ϵ and any value of μ. There is no cusp at $t = \pi/2$ if $\nu \geq -3/2$. A program for (A-7) follows, where ϵ, μ, ν, and B are arbitrary ($\epsilon \geq 0$, $\nu \geq -3/2$, $B \geq 0$).

```
10    Eps=.1          ε        ! Pattern Ve(u) via (A-7)
20    Mu=0            μ
30    Nu=-1.5         ν
40    Bc=4            B        ! B in (26)
50    OUTPUT 0;"Eps =";Eps;"  Mu = ";Mu;"  Nu = ";Nu;"  Bc =";Bc
60    DIM Ve(0:480)
70    COM U,Bc,M1,N1,M2,N2
80    M1=Mu-1
90    N1=Nu+1
100   M2=2*Mu-1
110   N2=2*Nu+3
120   Alpha=Mu+Nu+1
130   T=2*Eps-Eps^2
140   IF Eps=0 THEN 160
150   F1e=T^N1*FNInuxnu(N1,Bc*SQR(T))     ! for line 2 of (A-7)
160   E1=1-Eps
170   Te=E1^(2*Mu)
180   A=ASN(E1)
190   B=PI/2
200   FOR Iu=0 TO 480
210   U=Iu*PI/40
220   Sq=SQR(ABS(U*U-Bc*Bc))
230   IF U<=Bc THEN T1=FNInuxnu(Alpha,Sq)     ! line 1 of (A-7)
240   IF U>Bc THEN T1=FNJnuxnu(Alpha,Sq)
250   IF Eps>0 THEN 290
260   T3=T1
270   V=0
280   GOTO 480
290   T2=FNJnuxnu(Mu,U)-Te*FNJnuxnu(Mu,U*E1) ! for line 2 of (A-7)
300   T3=T1+F1e*T2
310   S=(FNS(A)+FNS(B))*.5
320   N=2
330   H=(B-A)*.5
340   F=(B-A)/3
350   Vo=9E99
360   T=0
370   FOR K=1 TO N-1 STEP 2
380   T=T+FNS(A+H*K)
390   NEXT K
400   S=S+T
410   V=(S+T)*F                    ! line 3 of (A-7)
420   IF ABS(V-Vo)<=ABS(V)*1E-5 THEN 480
430   Vo=V
440   N=N*2
450   H=H*.5
460   F=F*.5
470   GOTO 360
480   Ve(Iu)=T3-V                  ! Voltage Response (A-7)
490   PRINT Iu,Ve(Iu)
500   NEXT Iu
510   PLOTTER IS "GRAPHICS"
520   GRAPHICS
530   SCALE 0,480,-70,0
540   GRID 40,10
550   PENUP
560   FOR Iu=0 TO 480
570   PLOT Iu,20*LGT(ABS(Ve(Iu)/Ve(0)))
580   NEXT Iu
590   PENUP
600   END
610   !
```

```
620   DEF FNS(T)                    ! integrand of (A-7)
630   COM U,Bc,M1,N1,M2,N2
640   St=SIN(T)
650   Ct=COS(T)
660   T1=FNJnu.nu(M1,U*St)
670   T2=FNInu.nu(N1,Bc*Ct)
680   RETURN St^M2*Ct^N2*T1*T2
690   FNEND
700   !
710   DEF FNGamma(X) !  Gamma(X)  via HART, page 279, #5231
720   N=INT(X)
730   R=X-N
740   IF (N>0) OR (R<>0) THEN 770
750   PRINT "FNGamma(X) IS NOT DEFINED FOR X = ";X
760   STOP
770   IF R>0 THEN 800
780   Gamma2=1
790   GOTO 840
800   P=3.36954359131+R*(1.09850630453+R*(.142928007949+R*3.93013464186E-2))
810   P=43.9410209139+R*(22.9680800836+R*(12.3021698112+R*P))
820   Q=43.9410209191+R*(4.39050474596-R*(7.15075063299-R))
830   Gamma2=P/Q                    ! Gamma(2+R) for 0 < R < 1
840   IF N>2 THEN 880
850   IF N<2 THEN 930
860   Gamma=Gamma2
870   GOTO 980
880   Gamma=Gamma2
890   FOR K=1 TO N-2
900   Gamma=Gamma*(X-K)
910   NEXT K
920   GOTO 980
930   R=1
940   FOR K=0 TO 1-N
950   R=R*(X+K)
960   NEXT K
970   Gamma=Gamma2/R
980   RETURN Gamma
990   FNEND
1000  !
```

```
1010  DEF FNJnuxnu(Nu,X)         !  Jnu(X)/X^nu
1020  IF ABS(X)<1 THEN 1130
1030  A=.797884560803
1040  IF Nu=0 THEN RETURN FNJo(X)
1050  IF Nu=.5 THEN RETURN A*SIN(X)/X
1060  IF Nu=-.5 THEN RETURN A*COS(X)
1070  IF Nu=1 THEN RETURN FNJ1(X)/X
1080  IF Nu=-1 THEN RETURN -FNJ1(X)*X
1090  IF Nu=1.5 THEN RETURN A*(SIN(X)-X*COS(X))/X^3
1100  IF Nu=-1.5 THEN RETURN -A*(X*SIN(X)+COS(X))
1110  IF Nu=2 THEN RETURN (2*FNJ1(X)-X*FNJo(X))/X^3
1120  IF Nu=-2 THEN RETURN (2*FNJ1(X)-X*FNJo(X))*X
1130  A=Nu
1140  IF (INT(A)<>A) OR (A>=0) THEN 1160
1150  K=A=-Nu
1160  S=T=1/(2^A*FNGamma(A+1))
1170  R=-.25*X*X
1180  Big=ABS(S)
1190  FOR N=1 TO 100
1200  T=T*R/(N*(N+A))
1210  S=S+T
1220  Big=MAX(Big,ABS(S))
1230  IF ABS(T)<=1E-11*ABS(S) THEN 1270
1240  NEXT N
1250  PRINT "100 TERMS IN FNJnuxnu(Nu,X) AT ";Nu;X
1260  PAUSE
1270  D=12-LGT(ABS(Big/S))   ! NO. OF SIGNIF. DIGITS
1280  IF K>0 THEN S=S*(4*R)^K
1290  RETURN S
1300  FNEND
1310  !
1320  DEF FNJo(X)     ! Jo(X) via 9.4.1 & 9.4.3
1330  Y=ABS(X)
1340  IF Y>3 THEN 1390
1350  T=Y*Y/9
1360  Jo=.0444479-T*(.0039444-T*.00021)
1370  Jo=1-T*(2.2499997-T*(1.2656208-T*(.3163866-T*Jo)))
1380  GOTO 1450
1390  T=3/Y
1400  Jo=9.512E-5-T*(.00137237-T*(.00072805-T*.00014476))
1410  Jo=.79788456-T*(7.7E-7+T*(.00552740+T*Jo))
1420  S=.00262573-T*(.00054125+T*(.00029333-T*.00013558))
1430  T=Y-.78539816-T*(.04166397+T*(3.954E-5-T*S))
1440  Jo=Jo*COS(T)/SQR(Y)
1450  RETURN Jo
1460  FNEND
1470  !
1480  DEF FNJ1(X)     ! J1(X) via 9.4.4 & 9.4.6
1490  Y=ABS(X)
1500  IF Y>3 THEN 1550
1510  T=Y*Y/9
1520  J1=.00443319-T*(.00031761-T*.00001109)
1530  J1=X*(.5-T*(.56249985-T*(.21093573-T*(.03954289-T*J1))))
1540  GOTO 1610
1550  T=3/Y
1560  J1=.00017105-T*(.00249511-T*(.00113653-T*.00020033))
1570  J1=.79788456+T*(1.56E-6+T*(.01659667+T*J1))
1580  S=.00637879-T*(.00074348+T*(.00079824-T*.00029166))
1590  T=Y-2.35619449+T*(.12499612+T*(5.65E-5-T*S))
1600  J1=SGN(X)*J1*COS(T)/SQR(Y)
1610  RETURN J1
1620  FNEND
1630  !
```

```
1640  DEF FNInu(nu(Nu,X)          ! Inu(X) nu
1650  IF ABS(X)>1 THEN 1770
1660  A=.398942280401
1670  E=EXP(X)
1680  IF Nu=0 THEN RETURN FNIo(X)
1690  IF Nu=.5 THEN RETURN A*(E-1/E)/X
1700  IF Nu=-.5 THEN RETURN A*(E+1/E)
1710  IF Nu=1 THEN RETURN FNI1(X)/X
1720  IF Nu=-1 THEN RETURN FNI1(X)*X
1730  IF Nu=1.5 THEN RETURN A*((X-1)*E+(X+1)/E)/X^3
1740  IF Nu=-1.5 THEN RETURN A*((X-1)*E-(X+1)/E)
1750  IF Nu=2 THEN RETURN (X*FNIo(X)-2*FNI1(X))/X^3
1760  IF Nu=-2 THEN RETURN (X*FNIo(X)-2*FNI1(X))*X
1770  A=Nu
1780  IF (INT(A)<>A) OR (A>=0) THEN 1800
1790  K=A=-Nu
1800  S=T=1/(2^A*FNGamma(A+1))
1810  R=.25*X*X
1820  Big=ABS(S)
1830  FOR N=1 TO 100
1840  T=T*R/(N*(N+A))
1850  S=S+T
1860  Big=MAX(Big,ABS(S))
1870  IF ABS(T)<=1E-11*ABS(S) THEN 1910
1880  NEXT N
1890  PRINT "100 TERMS IN FNInu(nu(Nu,X) AT ";Nu;X
1900  PAUSE
1910  D=12-LGT(ABS(Big/S))    ! NO. OF SIGNIF. DIGITS
1920  IF K>0 THEN S=S*(4*R)^K
1930  RETURN S
1940  FNEND
1950  !
1960  DEF FNIo(X)      ! Io(X) via 9.8.1 & 9.8.2
1970  Y=ABS(X)
1980  IF Y>3.75 THEN 2030
1990  T=Y*Y/14.0625
2000  Io=.2659732+T*(.0360768+T*.0045813)
2010  Io=1+T*(3.5156229+T*(3.0899424+T*(1.2067492+T*Io)))
2020  GOTO 2070
2030  T=3.75/Y
2040  Io=.00916281-T*(.02057706-T*(.02635537-T*(.01647633-T*.00392377)))
2050  Io=.39894228+T*(.01328592+T*(.00225319-T*(.00157565-T*Io)))
2060  Io=Io*EXP(Y)/SQR(Y)
2070  RETURN Io
2080  FNEND
2090  !
2100  DEF FNI1(X)     ! I1(X) via 9.8.3 & 9.8.4
2110  Y=ABS(X)
2120  IF Y>3.75 THEN 2170
2130  T=Y*Y/14.0625
2140  I1=.02658733+T*(.00301532+T*.00032411)
2150  I1=X*(.5+T*(.87890594+T*(.51498869+T*(.15084934+T*I1))))
2160  GOTO 2210
2170  T=3.75/Y
2180  I1=.01031555-T*(.02282967-T*(.02895312-T*(.01787654-T*.00420059)))
2190  I1=.39894228-T*(.03988024+T*(.00362018-T*(.00163801-T*I1)))
2200  I1=SGN(X)*I1*EXP(Y)/SQR(Y)
2210  RETURN I1
2220  FNEND
```

Appendix B
PROGRAM FOR THE EVALUATION OF (59)

The integral of interest is, from (59),

$$v_\epsilon(u) = \int_0^1 dx\, \mathcal{J}_\mu(ux)\, x^{2\mu+1}\, w_\epsilon(x) \, . \qquad \text{(B-1)}$$

We approximate this integral by sampling at increment $\Delta = \frac{1}{N_x}$ and using Simpson's rule. There follows

$$v_\epsilon(u) \cong \frac{\Delta}{3} \sum_{n=0}^{N_x} \overline{w}_n\, \mathcal{J}_\mu(nu\Delta)\, (n\Delta)^{2\mu+1}\, w_\epsilon(n\Delta) \, , \qquad \text{(B-2)}$$

where

$$\left\{\overline{w}_n\right\} = 1,\ 4,\ 2,\ 4,\ \ldots,\ 4,\ 2,\ 4,\ 1 \, . \qquad \text{(B-3)}$$

The program below evaluates (B-2). Values of $x^{2\mu+1}\, w_\epsilon(x)$ are stored in array Xmuwe.

```
10    Eps=.1          ! TRANSITION PARAMETER ε
20    Mu=0            ! DIMENSIONALITY PARAMETER μ
30    Nu=-1.5         ! WEIGHTING PARAMETER ν
40    Bc=4            ! WEIGHTING PARAMETER B
50    Nx=2^8          ! NUMBER OF INCREMENTS IN X
60    Niu=240         ! NUMBER OF INCREMENTS IN U
70    DIM Xmuwe(0:1024),Ve(0:240)
80    REDIM Xmuwe(0:Nx),Ve(0:Niu)
90    Mu21=Mu*2+1
100   Nu1=Nu+1
110   Del=1/Nx
120   Xmuwe(Nx)=0
130   FOR I=0 TO Nx-1
140   X=I*Del
150   R2=1-X*X
160   Rs=SQR(R2)
170   Br=Bc*Rs
180   Iv=R2^Nu*FNInuxnu(Nu,Br)
190   IF X>1-Eps THEN 220
200   Xmuwe(I)=X^Mu21*Iv
210   GOTO 260
220   Iv1=R2^Nu1*FNInuxnu(Nu1,Br)
230   Ne=FNNeut(Eps,X)
240   Nep=FNNeutp(Eps,X)
250   Xmuwe(I)=X^Mu21*(Iv*Ne-Iv1*Nep/X)
260   NEXT I
270   FOR Iu=0 TO Niu
280   U=Iu/Niu*6*PI
290   Ud=U*Del
300   T=Xmuwe(0)+FNJnuxnu(Mu,U)*Xmuwe(Nx)
310   So=Se=0
320   FOR Ns=1 TO Nx-1 STEP 2
330   So=So+FNJnuxnu(Mu,Ns*Ud)*Xmuwe(Ns)
340   NEXT Ns
350   FOR Ns=2 TO Nx-2 STEP 2
360   Se=Se+FNJnuxnu(Mu,Ns*Ud)*Xmuwe(Ns)
370   NEXT Ns
380   Ve(Iu)=T+4*So+2*Se
390   PRINT Iu,Ve(Iu)
400   NEXT Iu
410   PLOTTER IS "GRAPHICS"
420   GRAPHICS
430   SCALE 0,Niu,-70,0
440   GRID Niu/6,10
450   PENUP
460   T=20*LGT(Ve(0))
470   FOR Iu=0 TO Niu
480   Y=20*LGT(ABS(Ve(Iu)))-T
490   PLOT Iu,Y
500   NEXT Iu
510   PENUP
520   END
530   !
```

```
540   DEF FNNeut(E,X)
550   IF X<=1-E THEN RETURN 1
560   Y=(X-1+.5*E)/E
570   T=Y*Y
580   T=15-T*(40-T*48)
590   RETURN .5-.125*Y*T
600   FNEND
610   !
620   DEF FNNeutp(E,X)
630   IF X<=1-E THEN RETURN 0
640   Y=(X-1+.5*E)/E
650   T=Y*Y
660   T=1-T*(8-16*T)
670   RETURN -1.875*T/E
680   FNEND
690   !
```

ALL OTHER FUNCTIONS ARE LISTED IN APPENDIX A.

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